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Uniform Position Properties and Hilbert Functions for Points on a Smooth Quadric*

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If X is a set of points of \mathbf{P}^3 with the uniform position property, we study the structure of the Hilbert function of X with special care in the case when X lies on a smooth quadric Q . In this particular situation we also investigate the behaviour of the Hilbert matrix of X when the uniform position property is enlarged to the divisors of Q . © 1992 Academic Press, Inc.

INTRODUCTION

It is well known that the generic hyperplane section of a reduced irreducible curve of \mathbf{P}^r has the uniform position property (see [ACGH, p. 116]). For points in \mathbf{P}^2 it was first proved that both generic plane sections of irreducible curves of \mathbf{P}^3 and points with the uniform position property (UPP for short) have a same kind of Hilbert function, i.e., its first difference is “of decreasing type” (see [MR₂]) and recently in [CO] it was shown that these two properties are indeed equivalent.

In studying points in \mathbf{P}^3 , one can obtain many different generalizations of the uniform position property. The most natural is to ask the points to have the uniform position with respect to the hypersurfaces (that is the usual UPP property) or, more strongly, to have the uniform position property with respect to the divisors of the irreducible surface of the smallest degree on which they lie (which we call for short strong-UPP property). Finally, according to what happens in \mathbf{P}^2 , one can ask them to be the section of an irreducible curve of \mathbf{P}^3 with a generic surface. We will see that these properties are not equivalent, not even for points on a smooth quadric. In this paper we want to study the shape of the Hilbert function for points in \mathbf{P}^3 lying on a smooth quadric with special care to the first two generalizations. We will deal with the third property in a forthcoming paper.

After preliminaries and notation in the first section we investigate some

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basic geometric properties for points on a smooth quadric having the UPP property.

In Section 2, as a consequence of a well-known Gieseker's lemma, (see [H]), we obtain some results on the first difference of the Hilbert function of points lying on a smooth quadric and we connect them to geometric properties for such points, thus generalizing analogous results obtained in [MR₁] for points in \mathbf{P}^2 .

The main results are in Section 3 in which we give the behaviour of the Hilbert function and the Hilbert matrix for points on a smooth quadric with the UPP property or with the strong-UPP property. More precisely, Theorem 3.5 states that for points X with the strong-UPP property the diagonal-difference of the Hilbert matrix decreases in each diagonal at least by 2 "as soon as possible," i.e., when it finishes to contain a minimal curve passing through X . When we apply this result to the Hilbert function we obtain (Corollary 3.6) that the first difference of it decreases at least by 2 after a possible stay constant for a while. We will see also (Theorem 3.9) that the same structure for the first difference of the Hilbert function is obtained, using a different approach, for points just having the UPP property.

NOTATION AND PRELIMINARIES

Throughout the paper we will consider projective spaces $\mathbf{P}_k^r = \mathbf{P}^r$ over an algebraically closed field k of characteristic zero.

For any closed subscheme $X \subset \mathbf{P}^3$ we will denote by

$$H(X, i) = \dim_k(S/I(X))_i = h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(i)) - h^0(\mathbf{P}^3, \mathcal{I}_X(i))$$

the Hilbert function of X , where $S = k[x_0, x_1, x_2, x_3]$, $I(X)$ is the homogeneous (saturated) ideal of S which defines X , $(S/I(X))_i$ means the i th graded component of the homogeneous coordinate ring of X , $\mathcal{O}_{\mathbf{P}^3}$ is the structural sheaf of \mathbf{P}^3 and \mathcal{I}_X is the ideal sheaf of X in \mathbf{P}^3 . When X is a zero-dimensional subscheme of \mathbf{P}^3 , many facts about $H(X, i)$ are known (see [R₁]). We will also use the following terminology:

$$\Delta H(X, i) = H(X, i) - H(X, i-1)$$

$$\begin{aligned} a(X) &= \min \left\{ i \in \mathfrak{N} \mid \Delta H(X, i) \neq \binom{i+2}{2} \right\} \\ &= \min \{ i \in \mathfrak{N} \mid H^0(\mathbf{P}^3, \mathcal{I}_X(i)) \neq 0 \} \end{aligned}$$

$$b(X) = \min \left\{ i \geq a(X) \mid \Delta H(X, i) < \binom{a(X)+2}{2} - 1 + a(X)(i - a(X)) \right\}$$

$$t(X) = \max \{ i \in \mathfrak{N} \mid \Delta H(X, i) \neq 0 \}.$$

Note that $a(X)$ is the smallest degree of a surface S containing X and $b(X)$ is the smallest degree of a surface containing X and not containing S . In this case, all these integers are well defined and, when there is no confusion, we often use a, b, t for $a(X), b(X), t(X)$.

When Q is a smooth quadric with an assigned embedding in \mathbf{P}^3 and X a closed subscheme of Q , \mathcal{I}'_X will denote the restriction of the ideal sheaf \mathcal{I}_X to Q . For any divisor $\mathcal{Q} \subset Q$ of type (a, b) , $\mathcal{O}_Q(a, b)$ will be the associated sheaf, and for any sheaf \mathcal{F} on Q , $\mathcal{F}(a, b)$ will just mean $\mathcal{F} \otimes \mathcal{O}_Q(a, b)$.

We recall that, if C is a curve of type (a, b) , $a \leq b$, on Q , it is

$$\Delta H(C, i) = \begin{cases} 2i + 1 & \text{for } 0 \leq i < b \\ a + b & i \geq b. \end{cases}$$

Furthermore, in this situation, we can define

$$M_X(i, j) = h^0(Q, \mathcal{O}_Q(i, j)) - h^0(Q, \mathcal{I}'_X(i, j))$$

the Hilbert matrix of X . For basic facts on the Hilbert matrix of X we refer to [GMR]. In particular, note that $H(X, i) = M_X(i, i)$. Because of the terminology used in [GMR], we will write

$$\Delta^d M_X(i, j) = M_X(i, j) - M_X(i-1, j-1),$$

i.e., the differences on the diagonals, not to be confused with the differences by columns or by rows.

For simplicity, we often omit the support of the sheaf on the cohomology groups. Here we use C.I. (a, b) for a one-dimensional subscheme of \mathbf{P}^3 which is a complete intersection of two surfaces of degrees a and b , and we use C.I. (a, b, c) for a zero-dimensional subscheme of \mathbf{P}^3 which is a complete intersection of three surfaces of degrees a, b , and c , respectively.

1. UNIFORM POSITION PROPERTIES

Let us start by stating definitions and elementary properties for points in uniform position.

DEFINITION 1.1. Let $X \subset \mathbf{P}^3$ be a set of points; we say that X has the uniform position property (UPP) if for every $X' \subset X$ it is

$$H(X', i) = \min\{|X'|, H(X, i)\} \quad \text{for all } i$$

(here $| \cdot |$ means “cardinality”).

Remark 1.2. It is easy to see that X has the UPP if and only if for every $X' \subset X$ and for every point $P \in X'$ it is

$$\Delta H(X' - \{P\}, i) = \begin{cases} \Delta H(X', i) & \text{if } i \neq t(X') \\ \Delta H(X', i) - 1 & \text{if } i = t(X') \end{cases}$$

PROPOSITION 1.3. *Let X be a set of points with the UPP property and $X' \subset X$ such that $\Delta H(X', i) < \Delta H(X, i)$. Then $\Delta H(X', j) = 0$ for every $j > i$.*

Proof. It is enough to use Remark 1.2. ■

COROLLARY 1.4. *Let $X \subset \mathbf{P}^3$ be a set of points with the UPP property lying on a unique C.I. $(2, b)$; then every component of such a complete intersection, say of type $(x, y) \neq (b, b)$, can contain at most $x + y + y^2$ points of X (assuming $x \leq y$).*

Proof. Let C' be the component of type (x, y) with $x \leq y$, contained in the complete intersection C of type (b, b) , then $\Delta H(C', i) \leq \Delta H(C, i)$. Now, since $\Delta H(C' \cap X, y) \leq x + y$ and $\Delta H(X, y) = \min\{2y + 1, 2b\}$, we have $\Delta H(C' \cap X, y) < \Delta H(X, y)$.

By Proposition 1.3 one obtains $\Delta H(C' \cap X, i) = 0$ for every $i > y$. So,

$$\begin{aligned} \Delta H(C' \cap X, i) &\leq \Delta H(C', i) & \text{for } i \leq y \\ \Delta H(C' \cap X, i) &= 0 & \text{for } i > y \end{aligned}$$

and, finally, $|C' \cap X| \leq x + y + y^2$. ■

We want to show that for points X with the uniform position property, there always exists an irreducible surface of degree $a(X)$ (the smallest degree of a surface containing X). Indeed, we prove a little more.

THEOREM 1.5. *Let $X \subset \mathbf{P}^3$ be a set of points with the UPP property and denote $|X| = \binom{d+3}{3} + h$, with $0 \leq h < \binom{d+3}{2}$ (note that such d and h are always uniquely determined), and $a = a(X)$. Then we have*

(1) *if either $a < d + 1$ or $a = d + 1$ and $h \geq 3$ then every element of $H^0(\mathcal{I}_X(a))$ is irreducible;*

(2) *if $a = d + 1$ and $0 \leq h \leq 2$, then there is an irreducible element in $H^0(\mathcal{I}_X(a))$.*

Proof. To prove (1) in case $a < d + 1$ it is sufficient to note that for every $\alpha, \beta \in \mathfrak{N}$ it is

$$\binom{\alpha+3}{3} + \binom{\beta+3}{3} < \binom{\alpha+\beta+3}{3}.$$

When $a = d + 1$ and $h \geq 3$, if $H^0(\mathcal{I}_X(a))$ contained a reducible element, say $F \cdot G$ with $\deg F = \alpha \leq \deg G = \beta$, $\alpha + \beta = a$, we would have, by the UPP hypothesis on X ,

$$\binom{\alpha+3}{3} - 1 + \binom{\beta+3}{3} - 1 \geq \binom{d+3}{3} + h = \binom{\alpha+\beta+2}{3} + h$$

from which we obtain $(\beta+1)(\beta+4) \geq \alpha\beta(\alpha+\beta+4) + 2h$.

This leads to $\alpha = 1$ and then $4 \geq 2h$, which is a contradiction.

Let us suppose now that (2) is not true. First note that $h^0(\mathcal{I}_X(a)) = \binom{d+3}{2} - h > 2(d+1)$ (unless the trivial case $d = 1$, $h = 2$ in which the conclusion (2) is trivially true). This says by Bertini's theorem that $H^0(\mathcal{I}_X(a))$ has a fixed component, say S with $\deg S = \alpha < a$. Denote $X' = X - (X \cap S)$. Then we would have

$$\begin{aligned} H(X', d+1-\alpha) &= \binom{d+1-\alpha+3}{3} - h^0(\mathcal{I}_{X'}(d+1)) \\ &= \binom{d+1-\alpha+3}{3} - \binom{d+3}{2} + h. \end{aligned}$$

Now, since $\binom{d+3}{2} - \binom{d+1-\alpha+2}{2} > 2 \geq h$, an easy computation shows that

$$\binom{d+1-\alpha+2}{3} > H(X', d+1-\alpha).$$

This just says that X' is contained on a surface of degree $d - \alpha$, therefore X should stay on a surface of degree $d < a$ and we obtain the desired contradiction. ■

Remark 1.6. Note that for points X with the UPP property belonging to a smooth quadric Q , despite the fact that there always exists an irreducible surface of degree

$$b = b(X) = \min\{i \geq 2 \mid \Delta H(X, i) < 2i + 1\},$$

through X , they do not necessarily stay on an irreducible curve of type (b, b) on Q , as we show in the following example.

On an irreducible curve of type $(1, 2)$ of Q take 7 "generic" points (here "generic" just means 4 by 4 not on a plane); then take two more points on a line of type $(1, 0)$, generic to the previous ones. By construction we obtain nine points X having the UPP property. In such a case it is $b = 2$ and the unique $(2, 2)$ passing through X is reducible.

In $[R_2]$ the structure of the Hilbert function of a set of points X on a

quadric Q was extensively studied. In particular, in the case when Q is smooth it was shown that if

$$b = b(X) = \min\{i \geq 2 \mid \Delta H(X, i) < 2i + 1\},$$

$$c = \min\{i \geq b \mid X \subseteq \text{C.I.}(2, b, i)\},$$

then $\Delta H(X, j) \geq \Delta H(X, j+1) + 2$ for every $j \geq c$. Of course, the above c is not determined by the Hilbert function of X , but from geometric properties of such a scheme. For points with the UPP property we can give a naive bound to the value of c . Thereafter we will give a more complete result on the structure of the Hilbert function for points with the UPP property.

PROPOSITION 1.7. *Let X be a set of points with the UPP property contained on a smooth quadric Q . Set*

$$c' = \min\{i \geq b = b(X) \mid \Delta H(X, i) < 2b\}.$$

If $c' \geq \frac{3}{2}b$ then X is contained in an irreducible C.I. $(2, b)$, i.e., $c = c'$.

Proof. First notice that the hypothesis $c' \geq \frac{3}{2}b$ implies as well that X is contained in a unique C.I. $(2, b)$. Let us suppose that such a complete intersection is reducible with irreducible components C_1, C_2, \dots, C_s and each C_i is of type (x_i, y_i) , with $x_i \leq y_i$. By Corollary 1.4 on every C_i there are at most $x_i + y_i + y_i^2$ points of X ; therefore,

$$|X| \leq \sum_{i=1}^s (x_i + y_i + y_i^2).$$

Now a simple computation shows that the right side takes its maximum value when $s=2$ and the two irreducible components are of the types $(1, b-1)$ and $(b-1, 1)$; in this case the value is $2b^2 - 2b + 2$. On the other hand, since

$$\Delta H(X, i) = \begin{cases} 2i + 1 & \text{for } 0 \leq i < b \\ 2b & \text{for } b \leq i < c' \\ \text{etc.,} \end{cases}$$

it is $|X| \geq b^2 + 2b(c' - b)$. So, we should have $b^2 + 2b(c' - b) \leq 2b^2 - 2b + 2$; from which we obtain $2bc' \leq 3b^2 - 2b + 2$, i.e., $c' \leq \frac{3}{2}b - 1 + 1/b < \frac{3}{2}b$, the required contradiction. ■

Using the example in Remark 1.6, one can see that for $c' < \frac{3}{2}b$ the C.I. $(2, b)$ can be reducible.

The UPP property as defined in 1.1 concerns the generic behaviour of

subsets of X only with respect to the linear systems of surfaces of \mathbf{P}^3 . But we can demand that the subsets of X are indistinguishable from each other with respect to the linear systems of the divisors of a surface in which they stay. For instance, one can prove that this stronger uniform property holds for the generic section of an irreducible curve of \mathbf{P}^3 with a smooth quadric. Note that this last geometric property (to be the section of an irreducible curve with a generic quadric) is indeed just a particular case of the strong uniformity about which we are talking. We will investigate these relationships and the structure of the related Hilbert functions in a forthcoming paper. Hence the following definition makes sense.

DEFINITION 1.8. Let $X \subset Q$ be a set of points on a smooth quadric, we will say that X has the strong-UPP property if for every $X' \subset X$ it is

$$M_{X'}(i, j) = \min\{|X'|, M_X(i, j)\} \quad \text{for all } (i, j) \in \mathfrak{N} \times \mathfrak{N}.$$

Remark 1.9. Since $M_X(i, i) = H(X, i)$, it follows that if X has the strong-UPP property, it has also the UPP property. Furthermore, as we described before, points which are a section of an irreducible curve with a generic quadric are in strong-UPP.

In the sequel we will use the partial order on $\mathfrak{N} \times \mathfrak{N}$ induced by the usual total order on \mathfrak{N} . Let us denote, for every $k \in \mathfrak{Z}$,

$$(i_k, j_k)_{X'} = \max\{(i, j) \in \mathfrak{N} \times \mathfrak{N} \mid j - i = k \text{ and } \Delta^d M_{X'}(i, j) \neq 0\}.$$

In analogy to what we stated in 1.2 we have

Remark 1.10. It is easy to see that X has the strong-UPP property if and only if for every $X' \subset X$, for every point $P \in X'$ and for every $k \in \mathfrak{Z}$,

$$\Delta^d M_{X' - \{P\}}(i, j) = \begin{cases} \Delta^d M_{X'}(i, j) & \text{for } (i, j) \neq (i_k, j_k)_{X'} \\ \Delta^d M_{X'}(i, j) - 1 & \text{for } (i, j) = (i_k, j_k)_{X'}. \end{cases}$$

An easy consequence of the definition of points X in strong-UPP is that if a curve C of Q of type $_{\mathbf{a}}(i, j)$ contains at least $(i+1)(j+1)$ points of X then C must contain X completely. Of course, Definitions 1.1 and 1.8 are not equivalent, as we can see in the following example.

EXAMPLE 1.11. Take seven points on a rational non-degenerate cubic of \mathbf{P}^3 and two more points not on it and such that all nine points are “generic” (this just means that every four of them are not on a plane and every seven of them give independent conditions to the quadrics). So, by construction, these nine points have the UPP property, but they do not

have the strong-UPP property, since there are seven of them lying on a curve of type $(1, 2)$ (and not all); this contradicts the previous remark.

2. HILBERT FUNCTIONS AND GEOMETRY FOR POINTS ON A SMOOTH QUADRIC

The following results are crucial in our study of the structure of the Hilbert function of points with one of the uniform properties defined in the previous section. Indeed, they give meaningful information on the geometry of points on a smooth quadric, according to their Hilbert matrix or Hilbert function.

Here, for simplicity, we make use of the following notation: for a set of points X on a smooth quadric Q ,

$$H^0(\mathcal{O}_Q, \mathcal{I}'_X(a, b)) = \Sigma_X(a, b) \quad \text{and} \quad H^0(\mathcal{O}_Q, \mathcal{O}_Q(a, b)) = \Sigma(a, b);$$

C will denote the generic plane section of Q ; the linear system that $\Sigma_X(a, b)$ cuts on C will be denoted by $\sigma_X(a, b)$ and its dimension by $r_X(a, b)$ (of course, $\deg \sigma_X(a, b) = a + b$); furthermore, we will denote by $(i_b, j_b) \in \mathfrak{N} \times \mathfrak{N}$ a pair such that $\Sigma_X(i_b, j_b) \neq \emptyset$ but $\Sigma_X(i, j) = \emptyset$ for every $(i, j) < (i_b, j_b)$ (note that such a pair is not unique); and finally, $(i_c, j_c) \in \mathfrak{N}$ will mean a pair such that $\Sigma_X(i_c, j_c)$ contains an irreducible element, but $\Sigma_X(i, j)$ consists of only reducible elements for every $(i, j) < (i_c, j_c)$.

LEMMA 2.1. *With the above notation, for every $(a, b) \geq (i_b, j_b)$,*

$$r_X(a, b) = a + b - \Delta^d M_X(a, b).$$

Proof. Since C is a generic plane section of Q , we have

$$\begin{aligned} r_X(a, b) &= \dim \Sigma_X(a, b) - \dim \Sigma'_X(a, b) - 1 \\ &= (a + 1)(b + 1) - M_X(a, b) - [ab - M_X(a - 1, b - 1)] - 1 \\ &= a + b - \Delta^d M_X(a, b), \end{aligned}$$

where $\Sigma'_X(a, b)$ denotes the linear system of the curves of type (a, b) of Q containing $X \cup C$. ■

COROLLARY 2.2. *For every $(a, b) \geq (i_b, j_b)$,*

$$\begin{aligned} \Delta^d M_X(a, b) &\geq \Delta^d M_X(a + 1, b); \\ \Delta^d M_X(a, b) &\geq \Delta^d M_X(a, b + 1). \end{aligned}$$

In particular,

$$\Delta^d M_X(a, b) \geq \Delta^d M_X(a+1, b+1).$$

Proof. Since $\sigma_X(a+1, b) \geq \sigma_X(a, b) + |P|$, for every $P \in Q$, we have $r_X(a+1, b) \geq r_X(a, b) + 1$; hence,

$$a+1+b-\Delta^d M_X(a+1, b) \geq a+b-\Delta^d M_X(a, b)+1$$

from which we obtain the conclusion. ■

Corollary 2.2 can be derived, by different methods, by Theorems 2.4 and 2.6 of [GMR].

LEMMA 2.3. *With the same notation, for every $(a, b) \geq (i_c, j_c)$ we have*

- (1) *if $\Delta^d M_X(a-1, b) > 0$ then $\Delta^d M_X(a-1, b) > \Delta^d M_X(a, b)$.*
- (2) *if $\Delta^d M_X(a, b-1) > 0$ then $\Delta^d M_X(a, b-1) > \Delta^d M_X(a, b)$.*

In particular, if $(a, b) > (i_c, j_c)$ and $\Delta^d M_X(a-1, b-1) > 1$ then

$$\Delta^d M_X(a-1, b-1) > \Delta^d M_X(a, b) + 1.$$

Proof. Of course, it is enough to prove (1); the other statements are similar or consequences.

If $(a-1, b)$ is not greater than or equal to (i_b, j_b) , it is $M_X(a-1, b) = a(b+1)$ and $M_X(a-2, b-1) = (a-1)b$; hence, $\Delta^d M_X(a-1, b) = a+b$. On the other hand, since $(a, b) \geq (i_c, j_c) \geq (i_b, j_b)$, it is

$$M_X(a, b) = (a+1)(b+1) - h$$

with

$$h > 0 \quad \text{and} \quad M_X(a-1, b-1) = ab;$$

therefore, $\Delta^d M_X(a, b) = a+b-h$. So, in this case,

$$\Delta^d M_X(a-1, b-1) > \Delta^d M_X(a, b)$$

trivially. Thus, we can suppose $(a-1, b) \geq (i_b, j_b)$. If $a > i_c$, from Lemma 2.1, we have

$$r_X(a-1, b) = a-1+b-\Delta^d M_X(a-1, b) < a-1+b.$$

This means, because of our choice of C and (a, b) , that $\sigma_X(a-1, b)$ has no fixed points and it is not complete; so, by the Gieseke lemma,

$$\dim[\sigma_X(a-1, b) + |P|] \geq r_X(a-1, b) + 2.$$

On the other hand, by $\sigma_X(a, b) \supseteq \sigma_X(a-1, b) + |P|$, we obtain

$$r_X(a, b) \geq r_X(a-1, b) + 2$$

and this implies

$$a + b - \Delta^d M_X(a, b) \geq a - 1 + b - \Delta^d M_X(a-1, b) + 2,$$

from which the conclusion of (1) follows.

If $a = i_c$, let us suppose $\Delta^d M_X(a-1, b) = \Delta^d M_X(a, b)$. In this situation, by a straight computation of the dimensions of the linear systems and the inclusions,

$$\sigma_X(a-1, b) \subseteq \sigma_X(a-1, b) + |P| \subseteq \sigma_X(a, b),$$

we should have $\sigma_X(a-1, b) + |P| = \sigma_X(a, b)$ and, by the Gieseker lemma, $\sigma_X(a-1, b)$ should be complete with $\Delta^d M_X(a-1, b) > 0$ fixed points. But, $\sigma_X(a-1, b) + |P|$ has the same fixed points as $\sigma_X(a-1, b)$; this gives us a contradiction, as $\sigma_X(a, b)$ has no fixed points. ■

Note that, whenever $(a, b) = (i_c, j_c)$ and, of course,

$$(a, b-1) \geq (i_b, j_b) \quad \text{or} \quad (a-1, b) \geq (i_b, j_b),$$

it is $\Delta^d M_X(a-1, b-1) > \Delta^d M_X(a, b)$.

Remark 2.4. We recall that $M_X(i, j)$ satisfies many numerical conditions, in particular, $M_X(i, j) = |X|$ for every $(i, j) \gg (0, 0)$; from this, it follows that $\Delta^d M_X(i, j) = 0$ for every $(i, j) \gg (0, 0)$ and then

$$\sum_{j-i=k} \Delta^d M_X(i, j) = |X| \quad \text{for any fixed } k \in \mathbb{Z}.$$

Hereafter in this section $Z_X(a, b)$ will denote the base-locus of the linear system $H^0(\mathcal{O}_Q, \mathcal{I}'_X(a, b))$. In the case when $\dim Z_X(a, b) = 0$, the next proposition gives bounds to its cardinality.

PROPOSITION 2.5. *Let $X \subset Q$ be a set of points on a smooth quadric such that $H^0(\mathcal{I}'_X(a, b))$ has no fixed components. Set*

$$\begin{aligned} h^0(\mathcal{I}'_X(a, b)) &= \alpha, & h^0(\mathcal{I}'_X(a-1, b-1)) &= \beta, \\ \Delta^d M_X(a, b) &= a + b + 1 - \alpha + \beta = \tau; \end{aligned}$$

then we have

$$(a+1)(b+1) - \alpha \leq |Z_X(a, b)| \leq ab - \beta + \lceil (\tau^2 + 2\tau)/4 \rceil$$

(here $\lceil x \rceil$ means the least integer bigger than or equal to x)

Proof. The first inequality is trivial. For the second one, put $Z = Z_X(a, b)$ and observe that the hypothesis on $H^0(\mathcal{F}'_X(a, b))$, the Lemma 2.3, and the equality $H^0(\mathcal{F}'_Z(a, b)) = H^0(\mathcal{F}'_X(a, b))$ imply

$$\begin{aligned}\tau &\geq \Delta^d M_Z(a, b) \geq \Delta^d M_Z(a+1, b+1) + 2 \\ &\geq \Delta^d M_Z(a+2, b+2) + 4 \geq \dots,\end{aligned}$$

so that

$$\sum_{i=0}^{\infty} \Delta^d M_Z(a+i, b+i) \leq \begin{cases} 1+3+\dots+\tau & \text{if } \tau \text{ is odd} \\ 2+4+\dots+\tau & \text{if } \tau \text{ is even.} \end{cases}$$

Therefore,

$$\sum_{i=0}^{\infty} \Delta^d M_Z(a+i, b+i) \leq \lceil (\tau^2 + 2\tau)/4 \rceil$$

and, by Remark 2.4, we obtain $|Z| = M_Z(a-1, b-1) + \sum_{i=0}^{\infty} \Delta^d M_Z(a+i, b+i)$ and this leads to the conclusion. ■

We will give bounds on the number of points which can stay on the fixed components of $H^0(\mathcal{F}'_X(a, b))$, when they exist.

If $X \subset Q$ is a set of points on a smooth quadric and Γ is a curve of Q of type (a, b) , we can give a first bound to $|X \cap \Gamma|$ just using the Hilbert matrix of X .

PROPOSITION 2.6. *Let $X \subset Q$ be a set of points on a smooth quadric and Γ be a curve on Q of type (a, b) . Then*

$$|X \cap \Gamma| \leq a(m+1) + b(n+1) - ab + \sum_{i=1}^{\infty} \Delta^d M_X(n+i, m+i)$$

for every $(n, m) \geq (a, b)$.

Proof. Let $X'' = X - X \cap \Gamma$; we have

$$\begin{aligned}h^0(\mathcal{F}'_X(n, m)) &\geq h^0(\mathcal{F}'_{X''}(n-a, m-b)) \geq (n-a+1)(m-b+1) - |X''| \\ &= (n-a+1)(m-b+1) - |X| + |X \cap \Gamma|.\end{aligned}$$

On the other hand,

$$\begin{aligned}h^0(\mathcal{F}'_X(n, m)) &= (n+1)(m+1) - M_X(n, m) \\ &= (n+1)(m+1) - \left(|X| - \sum_{i=1}^{\infty} \Delta^d M_X(n+i, m+i) \right).\end{aligned}$$

Hence,

$$|X \cap \Gamma| \leq (n+1)(m+1) - (n-a+1)(m-b+1) \\ + \sum_{i=1}^{\infty} \Delta^d M_X(n+i, m+i),$$

and the claim follows. ■

COROLLARY 2.7. *With the previous notation, for every $s \geq b$, it follows that*

$$|X \cap \Gamma| \leq H(\Gamma, s) + \sum_{i=1}^{\infty} \Delta H(X, s+i).$$

Proof. It is an easy consequence of Proposition 2.6 for $n = m = s$.

Remark 2.8. The best value of the bound in the previous corollary is obtained for $s = \min\{i \geq b \mid \Delta H(X, i) \leq a+b\}$.

LEMMA 2.9. *If $H^0(\mathcal{J}'_X(r, s))$ has fixed components of positive dimension, put Γ the set of such components, say of type (a, b) , and $X' = X \cap \Gamma$, $X'' = X - X'$; then we have*

(1) $H^0(\mathcal{J}'_{X''}(r-a, s-b))$ has no fixed components of positive dimension.

(2) $\Delta^d M_{X''}(i, j) = \Delta^d M_X(i+a, j+b) - a - b$ for every $(i, j) \leq (r-a, s-b)$. In particular, $\Delta^d M_{X''}(r, s) \geq a+b$.

(3) (i) $\Delta^d M_{X''}(r-a-1, s-b) > 0$ if and only if $\Delta^d M_X(r-1, s) > \Delta^d M_X(r, s)$.

(ii) $\Delta^d M_{X''}(r-a, s-b-1) > 0$ if and only if $\Delta^d M_X(r, s-1) > \Delta^d M_X(r, s)$.

(iii) If Γ is also the fixed component for $H^0(\mathcal{J}'_X(r-1, s))$ or $H^0(\mathcal{J}'_X(r, s-1))$, then $\Delta^d M_{X''}(r-a-1, s-b-1) > 1$ if and only if $\Delta^d M_X(r-1, s-1) > \Delta^d M_X(r, s) + 1$.

Proof. (1) is immediate.

$$\begin{aligned} (2) \quad M_{X''}(i, j) &= (i+1)(j+1) - h^0(\mathcal{J}'_{X''}(i, j)) \\ &= (i+1)(j+1) - h^0(\mathcal{J}'_X(i+a, j+b)) \\ &= (i+1)(j+1) - (i+a+1)(j+b+1) + M_X(i+a, j+b) \\ &= -a(j+1) - b(i+1) - ab + M_X(i+a, j+b). \end{aligned}$$

Similarly,

$$M_{X''}(i-1, j-1) = -aj - bi - ab + M_X(i+a-1, j+b-1)$$

for every $(i, j) \leq (r-a, s-b)$, from which the claim follows easily.

(3) (i) If $\Delta^d M_{X''}(r-a-1, s-b) > 0$ by Lemma 2.3 one obtains $\Delta^d M_{X''}(r-a-1, s-b) > \Delta^d M_{X''}(r-a, s-b)$; now, by using (2) we obtain the required result.

(ii) Can be proved similarly.

(iii) Can be proved easily by applying (1) and (2) to $H^0(\mathcal{F}'_X(r-1, s))$ or $H^0(\mathcal{F}'_X(r, s-1))$. ■

THEOREM 2.10. *Under the same hypotheses and notation of the previous lemma, we have*

$$N - \lceil (\gamma_+^2 + 2\gamma_+)/4 \rceil \leq |X'| \leq N,$$

where

$$N = a(s+1) + b(r+1) - ab + \sum_{i=1}^{\infty} \Delta^d M_X(r+i, s+i),$$

$$\gamma_+ = \max\{\Delta^d M_X(r, s) - a - b - 2, 0\}.$$

Proof. The inequality on the right side follows by Proposition 2.6. On the other hand, by Lemma 2.9(1) and Lemma 2.3, one obtains

$$\Delta^d M_{X''}(r-a, s-b) \geq \Delta^d M_{X''}(r-a+1, s-b+1) + 2 \geq \dots$$

Set $\alpha = \Delta^d M_{X''}(r-a+1, s-b+1)$, since

$$|X''| = M_{X''}(r-a, s-b) + \sum_{i=1}^{\infty} \Delta^d M_{X''}(r-a+i, s-b+i),$$

it follows that

$$|X''| \leq M_{X''}(r-a, s-b) + \lceil (\alpha^2 + 2\alpha)/4 \rceil.$$

But, by Lemma 2.9(2), $\Delta^d M_{X''}(r-a, s-b) = \Delta^d M_X(r, s) - a - b$. Hence, either $\alpha = 0$ or $\alpha \leq \Delta^d M_{X''}(r-a, s-b) - 2 = \Delta^d M_X(r, s) - a - b - 2$. It follows that

$$|X''| \leq M_{X''}(r-a, s-b) + \lceil (\gamma_+^2 + 2\gamma_+)/4 \rceil.$$

But, taking into account the hypothesis on $H^0(\mathcal{J}'_X(r, s))$, it is

$$h^0(\mathcal{J}'_X(r, s)) = h^0(\mathcal{J}'_{X''}(r-a, s-b)),$$

from which

$$(r+1)(s+1) - M_X(r, s) = (r-a+1)(s-b+1) - M_{X''}(r-a, s-b).$$

Therefore,

$$|X''| \leq M_X(r, s) - a(s+1) - b(r+1) + ab + \lceil (\gamma_+^2 + 2\gamma_+)/4 \rceil.$$

In conclusion,

$$\begin{aligned} |X'| &= |X| - |X''| \\ &\geq |X| - M_X(r, s) + a(s+1) + b(r+1) - ab - \lceil (\gamma_+^2 + 2\gamma_+)/4 \rceil \\ &= N - \lceil (\gamma_+^2 + 2\gamma_+)/4 \rceil. \quad \blacksquare \end{aligned}$$

COROLLARY 2.11. *If $\Delta^d M_X(r, s) \leq a + b + 2$, on the fixed component Γ there are exactly $a(s+1) + b(r+1) - ab + \sum_{i=1}^{\infty} \Delta^d M_X(r+i, s+i)$ points of X . In particular, if $H^0(\mathbf{P}^3, \mathcal{J}_X(s))$ has a fixed component of degree $\Delta H(X, s) - 2$, then*

$$|X \cap \Gamma| = H(\Gamma, s) + \sum_{i=1}^{\infty} \Delta H(X, s+i).$$

Proof. It follows directly from Theorem 2.10. \blacksquare

THEOREM 2.12. *Let $(a-1, b-1) \geq (i_b, j_b)$; if*

$$\Delta^d M_X(a-1, b-1) \leq \Delta^d M_X(a, b) + 1$$

then, either $H^0(\mathcal{J}'_X(a, b))$ has a fixed component Γ of type (x, y) such that $x + y = \Delta^d M_X(a, b)$ and

$$|X \cap \Gamma| = x(b+1) + y(a+1) - xy + \sum_{i=1}^{\infty} \Delta^d M_X(a+i, b+i)$$

or $H^0(\mathcal{J}'_X(a-1, b-1))$ has a fixed component Γ' of type (x', y') such that $x' + y' = \Delta^d M_X(a-1, b-1)$ and

$$|X \cap \Gamma'| = x'b + y'a - x'y' + \sum_{i=1}^{\infty} \Delta^d M_X(a-1+i, b-1+i).$$

Proof. The hypotheses and Corollary 2.2 lead to the following two cases:

- I. $\Delta^d M_X(a-1, b) = \Delta^d M_X(a, b)$ or
- II. $\Delta^d M_X(a-1, b) = \Delta^d M_X(a-1, b-1)$.

In the first case Lemma 2.3 implies that $H^0(\mathcal{J}'_X(a, b))$ has a fixed component Γ of type (x, y) . Then, setting $X'' = X - X \cap \Gamma$, by Lemma 2.9(2), one has

$$\Delta^d M_{X''}(a-x-1, b-y) = \Delta^d M_X(a-1, b) - x - y;$$

so, in such a case, by Lemma 2.9(3)(i), it will be $\Delta^d M_{X''}(a-x-1, b-y) = 0$, from which it follows that $x + y = \Delta^d M_X(a-1, b) = \Delta^d M_X(a, b)$. Then, by Corollary 2.11 it follows that

$$|X \cap \Gamma| = x(b+1) + y(a+1) - xy + \sum_{i=1}^{\infty} \Delta^d M_X(a+i, b+i).$$

In case II, by Lemma 2.3, it follows that $H^0(\mathcal{J}'_X(a-1, b))$ has a fixed component Γ' of type (x', y') which, by the same arguments as before, will be of degree $x' + y' = \Delta^d M_X(a-1, b) = \Delta^d M_X(a-1, b-1)$. But, $H^0(\mathcal{J}'_X(a-1, b-1))$ has a fixed component $\bar{\Gamma} \supseteq \Gamma'$ whose degree is less than or equal to $\Delta^d M_X(a-1, b-1)$, by Lemma 2.9(2) again. This implies $\bar{\Gamma} = \Gamma'$.

Obviously, if $\Delta^d M_X(a-1, b-1) = \Delta^d M_X(a, b)$, then both $H^0(\mathcal{J}'_X(a, b))$ and $H^0(\mathcal{J}'_X(a-1, b-1))$ have the same fixed component of type (x, y) with $x + y = \Delta^d M_X(a, b)$. ■

When we apply Theorem 2.12 to the Hilbert function of X we obtain

COROLLARY 2.13. *If $\Delta H(X, s-1) \leq \Delta H(X, s) + 1$, then either $H^0(\mathbf{P}^3, \mathcal{J}_X(s))$ has a fixed component Γ of degree $\Delta H(X, s)$ and*

$$|X \cap \Gamma| = H(\Gamma, s) + \sum_{i=1}^{\infty} \Delta H(X, s+i)$$

or $H^0(\mathbf{P}^3, \mathcal{J}_X(s-1))$ has a fixed component Γ' of degree $\Delta H(X, s-1)$ and

$$|X \cap \Gamma'| = H(\Gamma', s-1) + \sum_{i=1}^{\infty} \Delta H(X, s-1+i). \quad \blacksquare$$

3. STRUCTURE OF THE HILBERT FUNCTION

The results of the previous sections allow us to give conditions on the structure of the Hilbert function and the Hilbert matrix for points $X \subset Q$ having the UPP property or the strong-UPP property. We start the section by proving the following:

PROPOSITION 3.1. *Let $X \subset Q$ be a set of points having the strong-UPP property. Then any minimal system of curves through X of type (a, b) with $0 < a \leq b$, contains at least an irreducible element.*

Proof. Let us assume $H^0(\mathcal{I}'_X(a, b))$ is reducible. Then, by Bertini's theorem (see [Ha]), two possibilities can occur:

- (1) the linear system $H^0(\mathcal{I}'_X(a, b))$ has a fixed component Γ of type $(x, y) < (a, b)$;
- (2) the generic curve of $H^0(\mathcal{I}'_X(a, b))$ splits in $k \geq 2$ components of type $(r, s) < (a, b)$ moving on a pencil.

In case (1), set $X' = X \cap \Gamma$, $X'' = X - X'$, and $d = h^0(\mathcal{I}'_X(a, b)) = h^0(\mathcal{I}'_{X''}(a - x, b - y))$. Since $M_X(a, b) = (a + 1)(b + 1) - d$, it follows that $|X| \geq (a + 1)(b + 1) - d$. On the other hand, $|X'| \geq d$ by the minimality of the system $H^0(\mathcal{I}'_X(a, b))$. Then there exists a curve Γ_1 in $H^0(\mathcal{I}'_{X''}(a - x, b - y))$ through $d - 1$ points \bar{X}'' of X' . If $|X''| \geq (a - x + 1)(b - y + 1) - (d - 1)$, then $(a - x + 1)(b - y + 1)$ points should belong to Γ_1 ; this contradicts the strong-UPP property. Therefore,

$$\begin{aligned} |X'| &= |X| - |X''| \geq (a + 1)(b + 1) - d - (a - x + 1)(b - y + 1) + d \\ &= xy + x + y + [x(b - y) + y(a - x)] \geq xy + x + y + 1 \end{aligned}$$

as $a \neq 0$ and that again contradicts the strong-UPP hypothesis.

In case (2), if $(a, b) = k(r, s)$, then $|X| \leq k(rs + r + s)$; but

$$h^0(\mathcal{O}_Q(kr - 1, ks)) = k^2rs + kr > k(rs + r + s).$$

This means that there exists at least a curve of type $(a - 1, b)$ through X , contradicting the minimality of $H^0(\mathcal{I}'_X(a, b))$. ■

Remark 3.2. (i) In the above proposition we assumed $a \neq 0$ because, for $|X| = n$, the linear system $H^0(\mathcal{I}'_X(0, n))$ is minimal and contains a unique curve consisting of n lines.

(ii) In Proposition 3.1 we saw that any minimal system of curves through X must contain an irreducible element. Actually, a straight

computation shows that all the curves in such systems are irreducible, except in the following two cases:

(I) X consists of six generic points: $H^0(\mathcal{I}'_X(2, 2))$ is a minimal system and contains reducible elements (both union of two $(1, 1)$ -curves or union of a $(2, 1)$ -curve with a $(0, 1)$ -line).

(II) For $0 < a \leq b$ take $ab + b$ points in the following way: $ab + b - 1$ generic points on a curve Γ' of type $(a, b - 1)$ and a further point P on a line L of type $(0, 1)$. For a set X of points chosen in such a way the linear system $H^0(\mathcal{I}'_X(a, b))$ is minimal and contains the curve $\Gamma = \Gamma' \cup L$.

EXAMPLE 3.3. The Proposition 3.1 is not true for a set of points having only the UPP property; indeed, the current example shows a set X of points for which a minimal curve through them necessarily splits and, in addition, there exists a linear system of curves, not all containing the given minimal curve, but nevertheless having a fixed component of positive dimension.

Let us consider on a quadric Q an irreducible curve Γ of type $(1, 3)$ and take on it 11 generic points (here generic means 4 by 4 not on a plane, 9 by 9 not on another quadric). Let us take a further point $P \notin \Gamma$, generic with respect to the previous points. The set X of the 12 considered points has the UPP property by construction. If L is the $(1, 0)$ -line through P , the curve $\Gamma \cup L$ is the only $(2, 3)$ -curve through X ; therefore it is minimal. On the other hand, the linear system $H^0(\mathcal{I}'_X(2, 4))$ does not contain $\Gamma \cup L$, as a fixed component, but contains Γ , since the generic $(2, 4)$ -curve meets Γ in 11 points, rather than 10.

Remark 3.4. Note that, if X is a set with $|X| = n$ and for every $X' \subset X$ with $|X'| = (i + 1)(j + 1) < n$ no curves of type (i, j) contain X' , while for $(i + 1)(j + 1) \geq n$ it is $M_X(i, j) = n$, then X has the strong-UPP property. If such a property holds just for $i = j$, then X has the UPP property. We used this fact in the previous example to state that the 12 points had the UPP property.

We are now able to prove the following:

THEOREM 3.5. *Let $X \subset Q$ be a set of points having the strong-UPP property. Let (a_1, b_1) be a minimal element in $\{(i, j) \mid M_X(i, j) < (i + 1)(j + 1) \text{ and } i \neq 0\}$. Then, for every $(x, y) \geq (a_1, b_1)$, we have either*

(1) $M_X(x, y) = (x + 1)(y + 1) - (x - a_1 + 1)(y - b_1 + 1)$ and in such a case $h^0(\mathcal{I}'_X(a_1, b_1)) = 1$ and every curve in $H^0(\mathcal{I}'_X(x, y))$ contains the curve of type (a_1, b_1) ; or

(2) if $\Delta^d M_X(x, y) > 1$ then $\Delta^d M_X(x, y) \geq \Delta^d M_X(x + 1, y + 1) + 2$.

Proof. If $h^0(\mathcal{I}'_X(a_1, b_1)) > 1$ then, by Proposition 3.1, it follows that $(i_c, j_c) = (a_1, b_1)$ and we obtain the conclusion by applying Lemma 2.3.

If $h^0(\mathcal{I}'_X(a_1, b_1)) = 1$ and $M_X(x, y) \neq (x+1)(y+1) - (x-a_1+1)(y-b_1+1)$, then in $H^0(\mathcal{I}'_X(x, y))$ there exists a curve not containing the minimal curve (a_1, b_1) so that, by Proposition 3.1, again, Lemma 2.3 can be applied to (x, y) and then we obtain the conclusion. ■

When we apply the previous theorem to the Hilbert function, we obtain the following:

COROLLARY 3.6. *Let $X \subset Q$ be a set of points in strong-UPP. Then, either $\Delta H(X, -)$ decreases at least by two from b or there exists an integer $n > b$ such that*

$$\Delta H(X, i) = \begin{cases} \Delta H(X, b) = h & \text{for } b \leq i \leq n-1 \text{ with } b < h \leq 2b \\ \text{decreases at least by two from } n. \end{cases} \quad (*)$$

Proof. If $h = 2b$ then $n = c = \min\{i \geq b \mid \Delta H(X, i) < 2b\}$, since there are both a minimal (b, b) -curve through X , irreducible by Proposition 3.1, and a (c, c) -curve, not containing the mentioned (b, b) -curve. So, the corollary follows by the previous theorem.

If $h < 2b$, put $h = 2b - d$; then it will be $h^0(\mathcal{I}'_X(b, b)) = d + 1$. So that, either $H^0(\mathcal{I}'_X(b))$ has an irreducible element, in that case $\Delta H(X, -)$ decreases at least by two from b by the above theorem, or X is contained in a minimal irreducible curve Γ of type $(u, v) < (b, b)$, with $0 < u \leq v$ (if $u = 0$ this would imply the trivial case when X is on a line). Taking into account the hypothesis on b , it follows that $v = b$ and, by dimension arguments, $u = b - d = h - b$. Hence, $h > b$.

Put $n = \min\{i \geq b \mid \Delta H(X, i) < h\}$ and observe that $H^0(\mathcal{I}'_X(n))$ has no fixed components of positive dimension (since it does not contain Γ which was irreducible); therefore, the previous theorem can be applied starting from n . ■

It is worthwhile to note that, if $X \subset Q$ is a set of points in strong-UPP and $\Delta H(X, i) = 2b - d$ for $i = b, \dots, n-1$, with $n-1 > b$, then X lies on a unique irreducible curve of type $(b-d, b)$. So that, when $d \geq b$ or $\Delta H(X, b) \leq b$, then $\Delta H(X, -)$ must decrease by at least two from b .

We show that the same conclusion as the above corollary can be obtained by just assuming that X is only in UPP.

EXAMPLE 3.7. Of course, there exist sets of point with the strong-UPP property for which the case $n > b + 1$ occurs. It is sufficient to construct 19 points in strong-UPP on an irreducible $(1, 3)$ -curve. It is easy to check that the first difference of the Hilbert function for these points looks like

$$1 \quad 3 \quad 5 \quad 4 \quad 4 \quad 2 \quad 0 \rightarrow.$$

COUNTEREXAMPLE 3.8. Corollary 3.6 only gives a necessary condition on the first difference of the Hilbert function of sets of points in strong-UPP. In fact, there are sequences of integers which satisfy conditions (*) of the above corollary but which cannot be the first difference of the Hilbert function of any set of points in strong-UPP. Let us consider the sequence

$$\alpha: 1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 6 \quad 6 \quad 4 \quad 2 \quad 0 \rightarrow.$$

If there were on Q 43 points X in strong-UPP such that $\Delta H(X, -) = \alpha$, by Corollary 3.6, Corollary 2.13, and Proposition 3.1, they should stay on a unique irreducible $(1, 5)$ -curve and, on the other hand, there should exist a $(7, 7)$ -curve through X not containing the above $(1, 5)$ -curve. But two such curves meet in just 42 points.

THEOREM 3.9. *Let $X \subset Q$ be a set of points with the UPP property. Then, $\Delta H(X, -)$ satisfies conditions (*) of Corollary 3.6.*

Proof. If $\Delta H(X, -)$ decreases at least by two starting from b , the conclusion follows for $n = b$. Otherwise, there exist integers i such that $b \leq i \leq t-1$, for which $\Delta H(X, i) \leq \Delta H(X, i+1) + 1$. Denote

$$n-1 = \max\{i \mid b \leq i \leq t-1 \text{ and } \Delta H(X, i) \leq \Delta H(X, i+1) + 1\}$$

$$\text{and } h = \Delta H(X, n-1).$$

It will be enough to prove that $\Delta H(X, i) = h$ for every i such that $b \leq i \leq n-1$. Namely, if it were $\Delta H(X, \bar{i}) > h$ for $b \leq \bar{i} \leq n-1$, since by Theorem 2.12, $H^0(\mathcal{I}_X(n-1))$ must have a fixed component Γ of degree h or $h-1$, then $\Delta H(\Gamma, \bar{i}) < \Delta H(X, \bar{i})$. Therefore, $X - \Gamma \neq \emptyset$. Let $P \in X - \Gamma$ and $X' = X - \{P\}$. Then

$$\Delta H(X', i) = \begin{cases} \Delta H(X, i) & \text{for } i \neq r \\ \Delta H(X, i) - 1 & \text{for } i = r, \end{cases}$$

where r has to be a suitable integer less than or equal to $n-1$. In fact, if it were $r \geq n$, by Corollary 2.11,

$$|X' \cap \Gamma| < |X \cap \Gamma|;$$

but, this is not the case, since $P \notin \Gamma$. This is a contradiction because, by the UPP property and Remark 1.2, r should be equal to t . ■

Remark 3.10. Notice that, if X is in UPP, by Theorem 3.9, $\Delta H(X, -)$ decreases at least by two starting from n . But it can happen that $\Delta H(X, n-1) = \Delta H(X, n) + 1$, as shown by the following example:

Consider an irreducible $(2, 4)$ -curve Γ on Q and intersect Γ by the

generic surface S of degree 6. By a generalization of the uniform position lemma (cf. [ACGH]) the points $X = \Gamma \cap S$ have the UPP property and $|X| = 36$. On the other hand, using Theorem 3.9, one shows that $\Delta H(X, -)$ takes the form

$$1 \quad 3 \quad 5 \quad 7 \quad 6 \quad 6 \quad 5 \quad 3 \quad 0 \rightarrow.$$

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